MATHEMATICS

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AIM POINT
MATHEMATICS

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CIRCLES AND SYSTEM OF CIRCLES

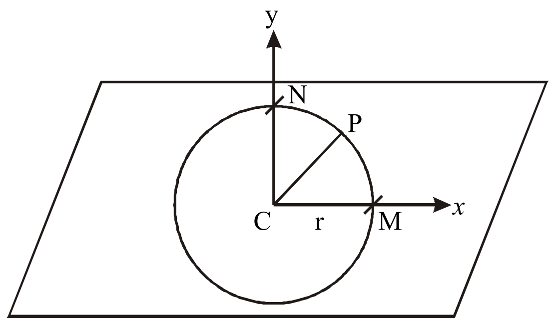
& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ Denfinition

A circle is defined as the locus of all such points in a plane, whihch remains at constant distance from a fixed point. Here the fixed point (C) is called the centre of the circle and the constant distance is called its radius.



✷ Equation of Circle in Different Forms

1. Genetal Equation of a Cirle

The general equation of second degree may represents a circle. If the coefficient of x^2 and coefficient of y^2 are identical and the coefficient of xy becomes zero.

ie, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ ...(i)

represents a circle, if

(a) a = b, ie, coefficient of x^2 = coefficient of y^2.

(b) h = 0, ie, coefficient of xy = 0,

then Eq. (i) reduces as,

$x^2 + y^2 + 2gx + 2fy + c = 0$, whose centre and radius are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$ respectively.

Nature of the Circle

If $g^2 + f^2 - c > 0$, then the radius of circle will be real and real circle is possible.

If $g^2 + f^2 - c = 0$, then the radius of circle is zero and the circle is called a point circle.

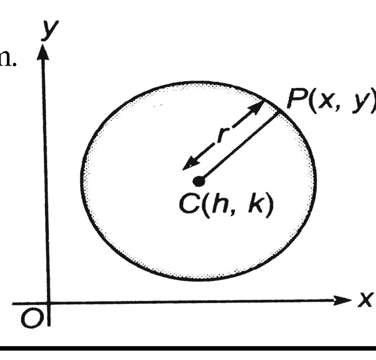
If $g^2 + f^2 - c < 0$, then the radius of circle will be imaginary, so in this case, no real circle is possible.

2. Equation of Circle in Standard Form

Let C(h, k) be the centre of circle and CP(= r) be the radius of circle, then equation of circle of circle is

$\therefore$ $(x - h)^2 + (y - k)^2 = r^2$ ....(i)

which is the equation of the circle in central form.



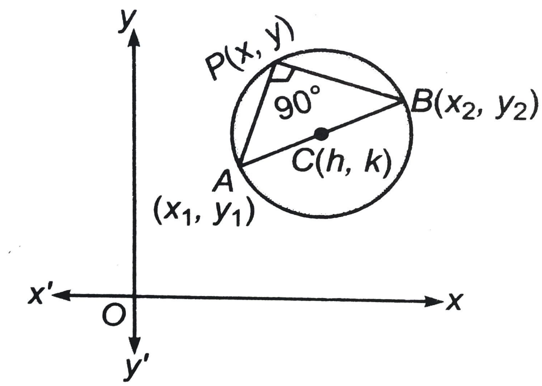
Now, if origin (0,0) be the centre of circle, then Eq. (i) becomes,

$$x^2 + y^2 = r^2.$$

3. Equation of Circle in DIameter Form

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of the given circle and let $P(x, y)$ be any point on the circle.

$\therefore$ $\angle ABC = 90^o.$



Slope of AP,

$$m_1 = \left(\frac{y - y_1}{x - x_1}\right)$$

and slope of BP,

$$m_2 = \left(\frac{y - y_2}{x - x_2}\right)$$

For perpendicular, $m_1 \cdot m_2 = -1$

$AP \cdot BP = -1$

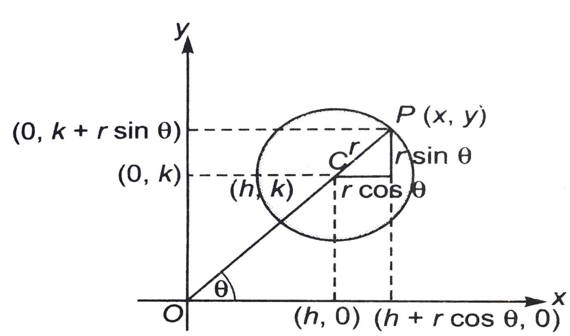
$$\Rightarrow \quad \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Which is the required equation of circle in diameter form.

4. Equation of circle in Parametric Form

A general point $P(x, y)$ on the circle in terms of parameters r and t can be expressed as



$$x = h + r\cos\theta$$

and $$y = k + r\sin\theta$$

$$x - h = r\cos\theta$$

$$y - k = r\sin\theta$$

On squaring and then adding, we get

$$(x - h)^2 + (y - k)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

which is the central equation of the circle satisfied by each fixed value of r (ie, radius) and varying θ.

If the circle is centered at origin, then the coordinates of centre are C(0, 0), then the parametric coordinates are $x = r$ cost and $y = r\sin\theta$.

5. Equation of Circle through Three Non-collinear Points

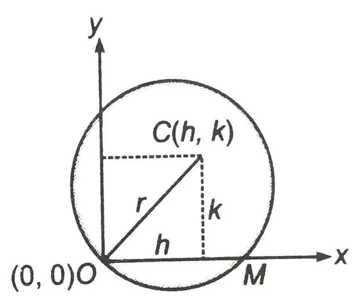
Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are three non-collinear points, then the eqution of circle through these three non-collinear points is given by

$$\begin{vmatrix} x^2+y^2 & x^2 & y^2 & 1 \\ x_1^2+y_1^2 & x_1^2 & y_1^2 & 1 \\ x_2^2+y_2^2 & x_2^2 & y_2^2 & 1 \\ x_3^2+y_3^2 & x_3^2 & y_3^2 & 1 \end{vmatrix} = 0$$

✷ **<u>Equation of Circle in Different Forms</u>**

Case I When the circle passes through the origin (0,0) Let the equation os circle be

$$(x - h)^2 + (y - k)^2 = r^2. \quad ....(i)$$



$\because$ It passes through origin (0,0).

$\therefore$ $$h^2 + k^2 = r^2 \quad ....(ii)$$

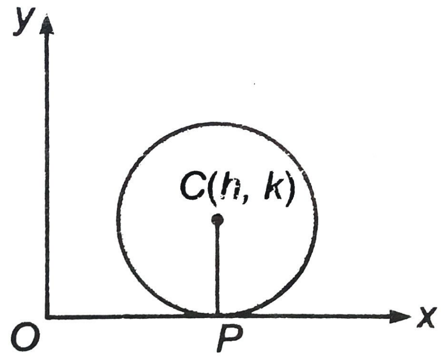
From Eqs. (i) and (ii), we get

$$(x - h)^2 + (y - k)^2 = h^2 + k^2$$

$\Rightarrow$ $$x^2 + h^2 - 2hx + y^2 - 2ky + k^2 = h^2 + k^2$$

$\Rightarrow$ $$x^2 + y^2 - 2hx - 2ky = 0$$

Case II When the circle touches abscissa (x-axis) Let the centre of circle be C(h, k) and it touches x- axis at point P, then the radius of circle is CP = k.

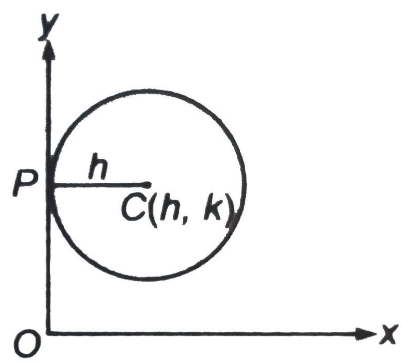


∴ Equation of circle is

$$(x - h)^2 + (y - k)^2 = (CP)^2 = k^2$$

or $$x^2 + y^2 - 2hx - 2ky + h^2 = 0$$

Case III When the circle touches ordinate (y-axis) Let the centre of circle be C(h, k) and it touches y-axis at point P with radius CP = h.

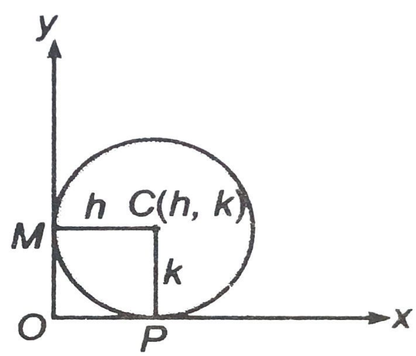


∴ Equation of circle is

$$(x - h)^2 + (y - k)^2 = (CP)^2 = h^2$$

or $$x^2 + y^2 - 2hx - 2ky + k^2 = 0.$$

Case IV When the circle touches both abscissa and ordinate (x-axis and y - axis) In this case $h = k = \alpha$.

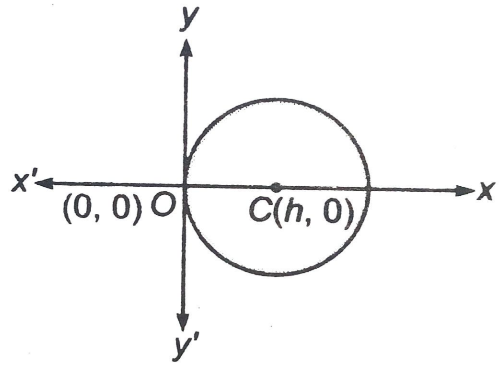


∴ Equation of circle is

$$(x - \alpha)^2 + (y - \alpha)^2 = \alpha^2$$

or $$x^2 + y^2 - 2\alpha x - 2\alpha y + \alpha^2 = 0.$$

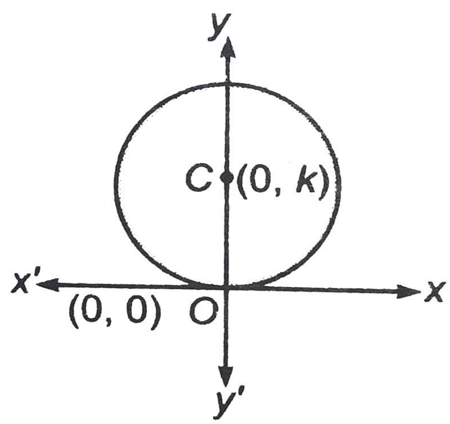
Case V When the circle passes through O(0, 0) and centre lines on abscissa (x-axis) In this case k = 0, then equation of circle is.



$$(x - h)^2 + (y - 0)^2 = h^2$$

$$\Rightarrow \quad x^2 + y^2 - 2hx = 0.$$

Case VI When the circle passes through O(0, 0) and centre lies on ordinate (y-axis).



In this case h = 0, then equation of circle is

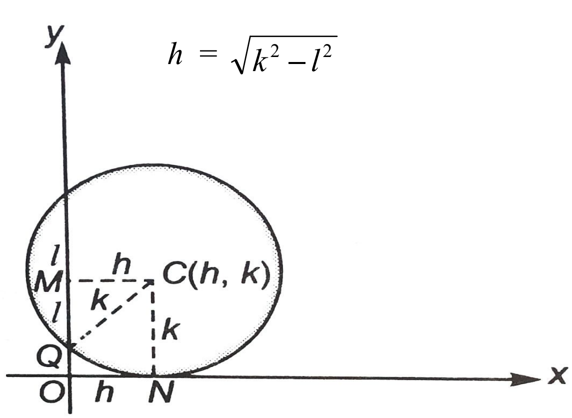
$$(x - 0)^2 + (y - k)^2 = k^2$$

$$\Rightarrow \quad x^2 + y^2 - 2ky = 0.$$

Case VII When the circle touches x-axis and cut off an intercepts on y-axis of length 2l.

In ΔCMQ, $\quad k^2 = h^2 + l^2$

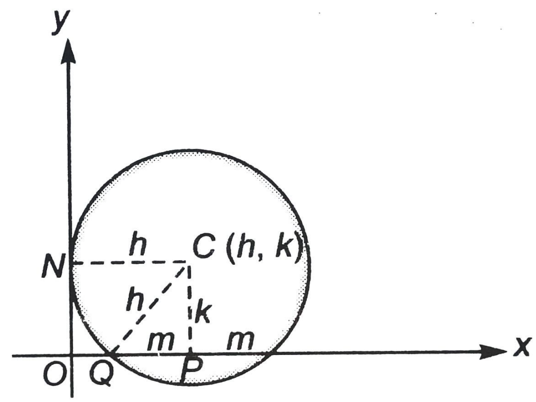
$$h = \sqrt{k^2 - l^2}$$



$\therefore$ Equation of circle is

$$[x-\sqrt{(k^2-l^2)}]+(y-k)^2=k^2$$

Case VIII When the circle touches y-axis and cut off an intercept on *x*-axis of length 2m.



In ΔCPQ, $h^2 = k^2 + m^2$

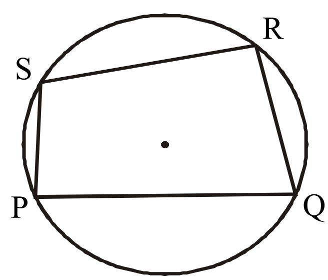
$k = \sqrt{h^2 - m^2}$

$\therefore$ Equation of circle is

$$(x-h)^2+[y-\sqrt{(h^2-m^2)}]=h^2$$

✷ Cyclic Quadrilateral

Quadrilateral, whose all four vertices lie on a circle is called a cyclic quadrilateral as shown in figure. The four vertices are said to be concyclic.



✷ Position of a Point with Respect to a Circle

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is the equation of the circle and $P(x_1, y_1)$ is any point in the plane of the circle, then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

Case I If $S_1 > 0$, then the point lies outside the circle.

Case II If $S_1 = 0$, then the point lies on the circle.

Case III If $S_1 < 0$, then the point lies inside the circle.

✷ Intersection of a Line and a Circle

Let $y = mx + c$ and $x^2 + y^2 = r^2$ be the equation of a line and a circle respectively.

$\therefore \quad x^2 + (mx + c)^2 = r^2 \quad (\because\ y = mx + c)$

$\Rightarrow \quad (1 + m^2)x^2 + 2mcx + (c^2 - r^2) = 0 \quad .....(i)$

Which is quadratic in 'x', then three cases arises.

Case I Roots of Eq. (i) are real and distinct, if $D = r^2(1 + m^2) - c^2 > 0$ ie, if $r > \frac{c}{\sqrt{1+m^2}}$.

Hence, the line meets the circle at two distinct points.

Case II Roots of Eq. (i) are coincidet, if

$$D = r^2(1 + m^2) - c^2 = 0 \text{ ie, if } r = \frac{c}{\sqrt{1+m^2}}.$$

Hence, the line touches the circle.

Case III Root of Eq. (i) are imaginary, if

$$D = r^2(1 + m^2) - c^2 < 0 \text{ ie, if } r < \frac{c}{\sqrt{1+m^2}}.$$

Hence, the line will not intersect the circle at all.

✷ Tangent to a Circle

A straight line is a tangent to a circle, if the distance of the centre from the liine equals the radius.

Different Form of the Equation of Tangents

Point Form

The equation of tangent at the point $P(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is

$$T \equiv xx_1 + yy_1 = r^2$$

Also, the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point $P(x_1, y_1)$ is

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Slope Form

The equation of a tangent of slope m to the circle $x^2 + y^2 = r^2$ is $T \equiv y = mx \pm r\sqrt{(1+m^2)}$ and the coordinates of the point of contact are $\left(\pm\frac{rm}{\sqrt{1+m^2}}, \mp\frac{r}{\sqrt{1+m^2}}\right)$.

Parametric Form

The equation of tangent ot the circle $x^2 + y^2 = r^2$ at the point $(r\cos\theta, r\sin\theta)$ is $T \equiv x\cos\theta + y\sin\theta = r$.

Length of the Tangent

The length of the tangent drawn from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.

Condition of Tangency

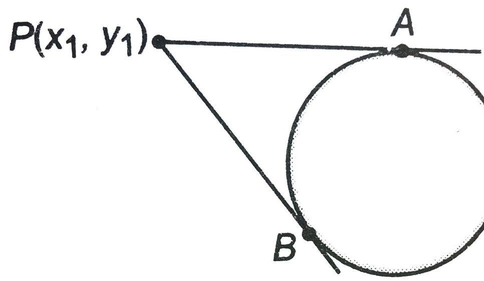
1. The line y = mx + c touches the circle $x^2 + y^2 = r^2$, iff $c = \pm r\sqrt{1+m^2}$.
2. The line lx + mx + n = 0 will touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, iff

   $(l^2 + m^2)(g^2 + f^2 - c) = (lg + mf - n)^2$.

Equation of Pair of Tangents

Let $P(x_1, y_1)$ be any point lying outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of pair of tangents represented by PA and PB is.

$$SS_1 = T^2$$



Where, $S = x^2 + y^2 + 2gx + 2fy + c$

$S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

and $T = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

Similarly, Equation of pair of tangents at the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is

$SS_1 = T^2$

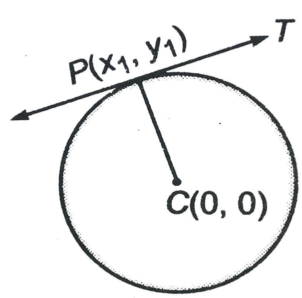
Where, $S = x^2 + y^2 - r^2$

$S_1 = x_1^2 + y_1^2 - r^2$

and $T = xx_1 + yy_1 - r^2$.

✶ **Normarl to a Circle**

The normal to a circle at any point is a straight line which is perpendicular to the tangent at that point and always passes throught the centre of the circle.



Equation of Normal

1. Equation of normal to the circle

$x^2 + y^2 = r^2$ at the point (x_1, y_1) is $\frac{x}{x_1} = \frac{y}{y_1}$.

2. Equation of normal to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

at the point (x_1, y_1) is

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}.$$

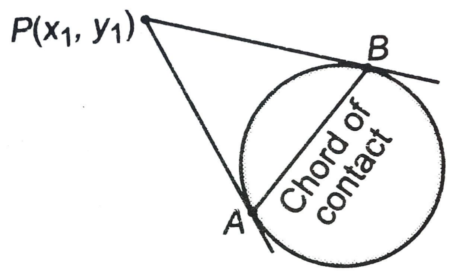
3. Equation of normal to the circle $x^2 + y^2 = r^2$ at $(r\cos\theta, r\sin\theta)$ is $\frac{x}{r\cos\theta} = \frac{y}{r\sin\theta}$ or $y = x\tan\theta$.

✷ **Chord of Contact**

From any external point, two tangents can be drawn to a given circle. The chord joining the points of contact of the two tangents is called the chord of contact of tangents.

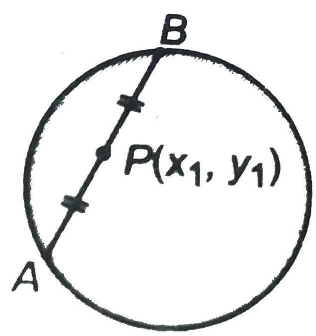
The equation of the chord of contact of tangents drawn from the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ and to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is.

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$



✷ **Equation of the Chord Bisected at a given Point**

Let $P(x_1, y_1)$ is the mid point of chord AB of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of circle besected at the point $P(x_1, y_1)$ is



$$T = S_1$$

$\Rightarrow \quad xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1$

Similarly, for the circle $x^2 + y^2 = r^2$, equation of chord is

$$T = S_1$$

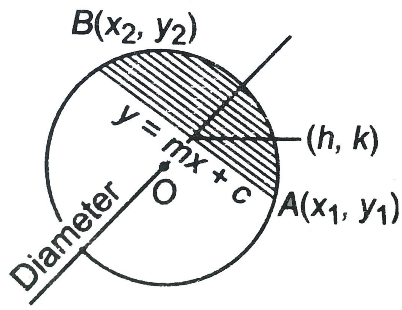
$\Rightarrow \quad xx_1 + yy_1 - r^2 = x_1^2 + y_1^2 - r^2.$

✷ **Diameter of a Circle**

The locus of the middle points of a system of paralel chords of a circle is called a diameter of the circle.

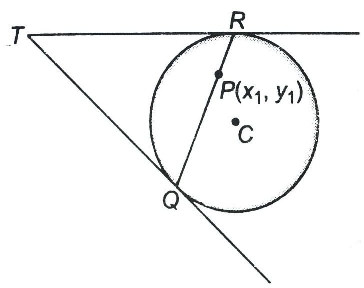
The equation of the diameter besecting parallel chords $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is.

$x + my = 0$.



✱ Pole and Polar

If through a point $P(x_1, y_1)$ (within or without a circle) there can be drawn any straight line to meet the given circle at Q and R, then locus of the point of intersection of the tangents at Q and R is called the polar of P and P is called the pole of the polar.



The Polar of point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$ and to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Conjugate Points

Two points A and B are conjugate points with respect to a given circle if each lies on the polar of the other with respect to the circle.

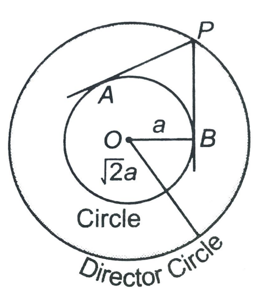
Conjugate Lines

If two lines be such that the pole of one lies on the other, then they are called conjugate lines with resect to the given circle.

✱ Director Circle

The locus of the point of intersection of two perpendicular tangents to a given circle is known as director circle.

If the equation of circle is $x^2 + y^2 = a^2$, then the equation of the direction circle to this circle is $x^2 + y^2 = 2a^2$.

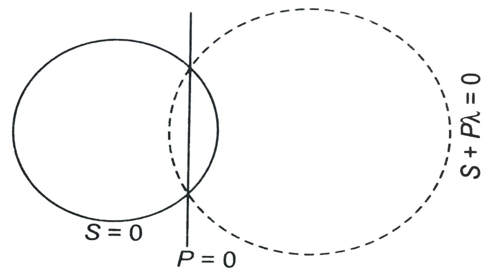


✷ Concentric Circles

Two circles having the same centre but centre but different radii r1 and r2 respectively, are called concentric circles. Thus, the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2gx + 2fy + \mu = 0$ are concentric circles. Therefore, the equations of concentric circles differ only in constat term.

✷ Family of Circles

1. Family of circles passing through the point of intersection of line $P \equiv lx + my + n = 0$ and circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $S + \lambda p = 0$.

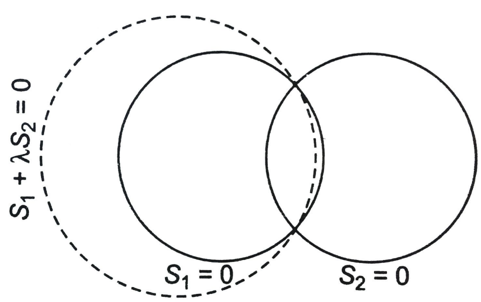


2. Family of circle passing through the points of intersection of two given circles.

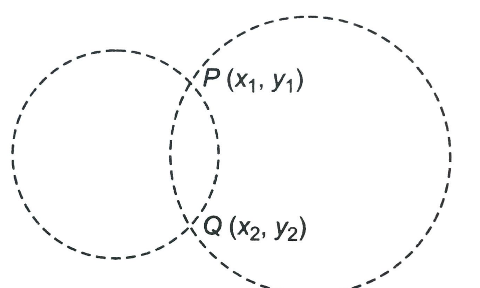
$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0,$$

is $$S_1 + \lambda S_2 = 0.$$



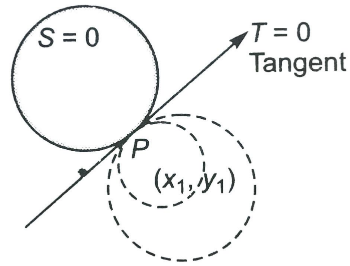
3. Equation of family of circles passing through two points (x_1, y_1) and (x_2, y_2) is



$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

4. The equation of the family of circles touching the circle

   $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at $P(x_1, y_1)$ is.



$$x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$$

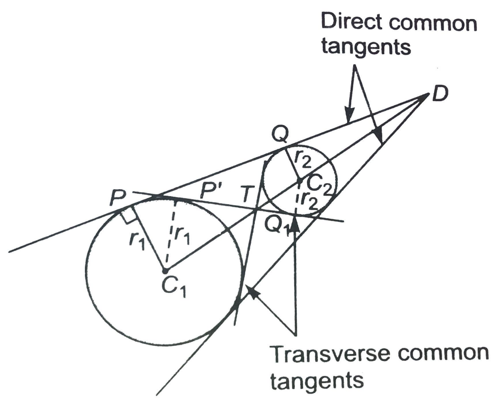
or $$S + \lambda T = 0$$

Where, T = 0 is the equation of the tangent to S = 0 at (x_1, y_1) and $\lambda \in R$.

✷ **Common Tangents to Two Circles**

1. Circles Neither Touching Nor Intersecting

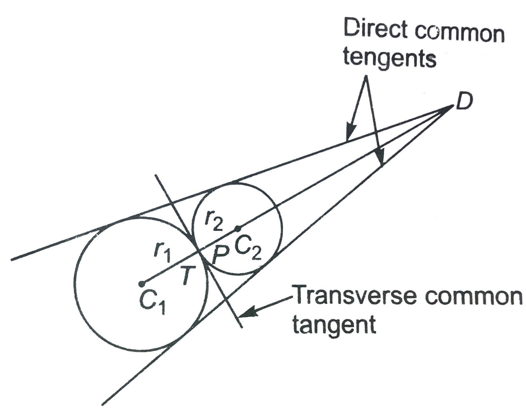
If the distance between the centers is more then the sum of their radii, then geometrically the two circles neither touch nor intersect each other. Mathematically, $|C_1C_2| > r_1 + r_2$.



In such a case there are two direct common tangents and two transverse common tangents.

2. Circles Touching Each Other Externally

If the distance between the centres equals the sum of their radii, then geometrically the two circle touch each other externaly. The point P where the two circles touch each other is called the point of contact. P divides the line joining centres internally in the ratio of their radii ie, $\dfrac{C_1P}{C_2P} = \dfrac{r_1}{r_2}$.



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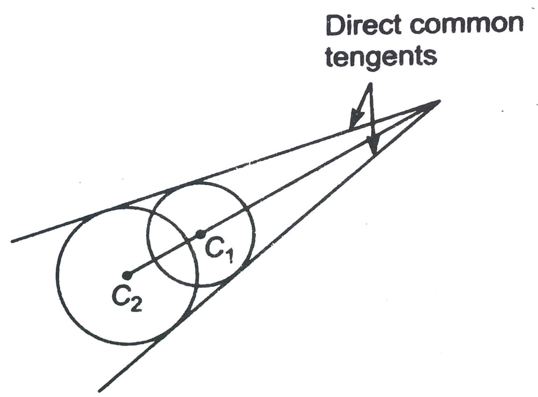
Mathematically, $|C_1C_2| = r_1 + r_2$.

In such a case, there are 3 common tangents to two touching circles.

3. Circles Intersecting Each other

If the distance between centres is less then the sum of their radii, then geometrically, the two circles intersect each other.



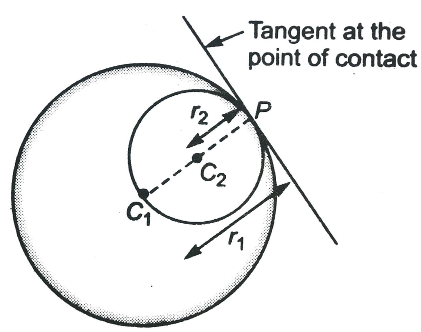
Mathematically, $|C_1C_2| < r_1 + r_2$.

In such a case, there are two direct common tangents.

4. Circles Touching Each Other Internally

If the distance between centres euqals the difference of the radii, the geometrically, the two circles touch each other internally. The point of contact P divides the line joining the centres externally in the ratio of their radii, ie.

$$\frac{C_1P}{C_2P} = \frac{r_1}{r_2}.$$

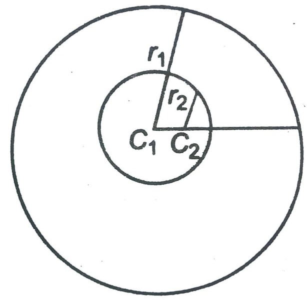


Mathematically, $|C_1C_2| = |r_1 - r_2|$.

In such a case, there is one common tangent to two circles.

5. One Circle Lies Inside Another

If distance between centres is less then the difference of the radii, then geometrically, one circle lies inside the another.

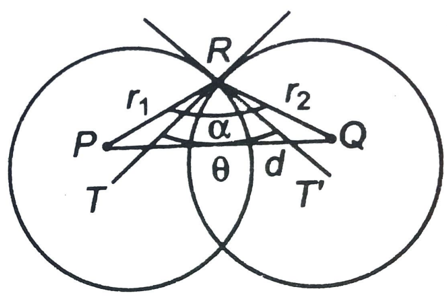


Mathematically, $|C_1C_2| < |r_1 - r_2|$.

In such a case, there is one common tangent to two circles.

✷ Angle of Intersection of Two Circles

The angle of intersection of two intersecting circles is the angle between their tangents at the points of intersection,



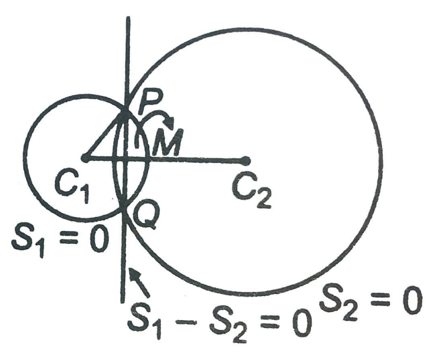
Where, d is distance between centres of the circles.

Condition of Orthogonality

If the angle between the circle is 90^o, then the circles are said to be orthogonal circle. Let $S_1 = 0$ and $S_2 = 0$ be any two circles, then condition of orthogonality is $2(g_1g_2 + f_1f_2) = c_1 + c_2$.

✷ Common Chord of Two Circles

The common chord joining the point of intersection of two given circles is called their common chord.



The equation of the common chord of two circles.

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0,$$

is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

ie, $S_1 - S_2 = 0$

Length of Common Chord

Length of Common chord = PQ = 2(PM) = $2\sqrt{\{(C_1P)^2 - (C_1M)^2\}}$

where, C_1P = radius of circle $S_1 = 0$

and C_1M = length of Perpendicular from C_1 on common chord PQ.

✱ **<u>Radical Axis</u>**

The radical axis of two circles is the locus of a point which moves in such a way that the length of the tangents drawn from it to the circles are equal. The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$.

Properties of Radical Axis

(a) The equation of radical axis and the common chord of two circle are identical.

(b) Radical axis is perpencicular to the straight line which joins the centres of the circles.

(c) The radical axis bisects common tangents of two circles.

(d) The radical axis of three circles taken in pairs are concurrents.

(e) **Radical centre** The point of intersection of radical axes of three circles whose centres are non-collinear, taken in pairs, is called their radical centre. The circle with centre at the radical center and radius equal to the length of the tangent drawn from it to one of the circles, intersects all the tree circles orthogonally.

Note :

- The length of intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on x-axis is $2\sqrt{g^2 - c}$ and y-axis $2\sqrt{f^2 - c}$.
- Length of perpendicular from (0, 0) on $y = mx + c$ is $\frac{c}{\sqrt{1+m^2}}$.
- If $y = mx + c$ is the tangent to the circle $x^2 + y^2 = r^2$, then the coordinates of point of contact are $\left(-\frac{mr^2}{c}, \frac{r^2}{c}\right)$.
- If the line $ax + by + c = 0$ is the tangent to the circle $x^2 + y^2 = r^2$, then the coordinates of point of contatct are $\left(-\frac{ar^2}{c}, \frac{br^2}{c}\right)$.
- The length of the intercept cut off from the line y = mx + c by the circle x2 + y2 = r2 is $2\sqrt{\frac{a^2(1+m^2)-c^2}{1+m^2}}$.
- Normal always passes through the centre of the circle.
- The diameter correspoinding to a system of parallel chords always passes through the centre of the circle and perpendicular to the parallel chords.
- Coordinates of radical center can be found by solving the equations
  $$S_1 = S_2 = S_3 = 0$$
- A system of circles (or family of circles) is said to be coaxial system of circles if every pair of the circles in the system has the same radical axis.